

# NON-PERTURBATIVE RESULTS FOR HIGH- $T$ QCD

R. JACKIW\*

*Center for Theoretical Physics, Department of Physics and Laboratory for Nuclear Science  
Massachusetts Institute of Technology  
Cambridge, MA 02139 U.S.A.*

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These days, as high energy particle colliders become unavailable for testing speculative theoretical ideas, physicists are looking to other environments that may provide extreme conditions where theory confronts physical reality. One such circumstance may arise at high temperature  $T$ , which perhaps can be attained in heavy ion collisions or in astrophysical settings. It is natural therefore to examine the high-temperature behavior of the standard model, and here I shall report on recent progress in constructing the high- $T$  limit of QCD.

In studying a field theory at finite temperature, the simplest approach is the so-called imaginary-time formalism. We continue time to the imaginary interval  $[0, 1/iT]$  and consider bosonic (fermionic) fields to be periodic (anti-periodic) on that interval. Perturbative calculations are performed by the usual Feynman rules as at zero temperature, except that in the conjugate energy-momentum, Fourier-transformed space, the energy variable  $p^0$  (conjugate to the periodic time variable) becomes discrete — it is  $2\pi nT$ , ( $n$  integer) for bosons. From this one immediately sees that at high temperature — in the limiting case, at infinite temperature — the time direction disappears, because the temporal interval shrinks to zero. Only zero-energy processes survive, since “non-vanishing energy” necessarily means high energy owing to the discreteness of the energy variable  $p^0 \sim 2\pi nT$ , and therefore all modes with  $n \neq 0$  decouple at large  $T$ . In this way

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a Euclidean three-dimensional field theory becomes effective at high temperatures and describes essentially static processes.[1]

Let me repeat in greater detail. Finite- $T$ , imaginary-time perturbation theory makes use of conventional diagrammatic analysis in “momentum” space, with modified “energy” variables, as indicated above. Specifically a spinless boson propagator is

$$D(p) = \frac{i}{p_0^2 - \mathbf{p}^2 - m^2} \quad p_0 = i\pi(2n)T \quad (1)$$

while a spin- $\frac{1}{2}$  fermion propagator reads

$$S(p) = \frac{i}{\gamma^0 p_0 - \boldsymbol{\gamma} \cdot \mathbf{p} - m} \quad p_0 = i\pi(2n+1)T \quad (2)$$

The zero-temperature integration measure  $\int \frac{d^4 p}{(2\pi)^4}$  becomes replaced by  $iT \sum_{n=-\infty}^{\infty} \int \frac{d^3 p}{(2\pi)^3}$ . Thus it is seen that Bose exchange between two  $O(g)$  vertices contributes  $iT \sum_{n=-\infty}^{\infty} \int \frac{d^3 p}{(2\pi)^3} g \frac{i}{-4\pi^2 n^2 T^2 - \mathbf{p}^2 - m^2} g$  where  $g$  is the coupling strength. In the large  $T$  limit, all  $n \neq 0$  terms (formally) vanish and only the  $n = 0$  term survives. One is left with  $\int \frac{d^3 p}{(2\pi)^3} g \sqrt{T} \frac{1}{\mathbf{p}^2 + m^2} g \sqrt{T}$ . This is a Bose exchange graph in a Euclidian 3-dimensional theory, with effective coupling  $g\sqrt{T}$ . Similar reasoning leads to the conclusion that fermions decouple at large  $T$ .

While all this is quick and simple, it may be physically inadequate. First of all, frequently one is interested in non-static processes in real time, so complicated analytic continuation from imaginary time needs to be made before passing to the high- $T$  limit, which in imaginary time describes only static processes. Also one may wish to study amplitudes where the real external energy is neither large nor zero, even though virtual internal energies are high.

Another reason that the above may be inadequate emerges when we consider massless fields (such as those that occur in QCD). We have seen that the  $n = 0$  mode leaves a propagator that behaves as  $\frac{1}{\mathbf{p}^2}$  when mass vanishes, and a phase space of  $d^3 p$ . It is well known that this kind of kinematics at low momenta leads to infrared divergences in perturbation theory even for off-mass-shell amplitudes — Green’s functions in massless Bosonic field theories possess infrared divergences in naive perturbation theory.[2] Since physical QCD does not suffer from off-mass-shell infrared divergences, perturbation theory must be resummed.

Thus the formal arguments for the emergence of a 3-dimensional theory at high- $T$  need be re-examined for QCD. Nevertheless, even if unreliable, the arguments alert us to the possibility that 3-dimensional field theoretic structures may emerge in the high- $T$  regime. Indeed this occurs, although not in a direct, straightforward fashion; this will be demonstrated presently.

Here is a graphical argument to the same end discussed above: *viz.* the need to resum perturbation theory. Consider a one-loop amplitude  $\Pi_1(p)$ ,

$$\Pi_1(p) \equiv \int dk I_1(p, k) ,$$

given by the graph in the figure.

$$\begin{aligned} \Pi_1(p) &= \text{Diagram of a one-loop bubble with external momentum } p \text{ and internal momentum } k \\ &\equiv \int dk I_1(p, k) \end{aligned}$$

Compare this to a two-loop amplitude  $\Pi_2(p)$ ,

$$\Pi_2(p) \equiv \int dk I_2(p, k) ,$$

in which  $\Pi_1$  is an insertion, as in the figure below.

$$\begin{aligned} \Pi_2(p) &= \text{Diagram: A horizontal line with momentum p enters from the left, splits into two paths that meet at a vertex on the right. The upper path is a straight line with momentum k. The lower path is a loop with momentum k, containing a circle with coupling g. The vertex on the right is connected to another horizontal line. Vertical dashed lines mark the vertices.} \\ &\equiv \int dk I_2(p, k) \end{aligned}$$

Following Pisarski,[3] we estimate the relative importance of  $\Pi_2$  to  $\Pi_1$  by the ratio of their integrands,

$$\frac{\Pi_2}{\Pi_1} \sim \frac{I_2}{I_1} = g^2 \frac{\Pi_1(k)}{k^2} ,$$

Here  $g$  is the coupling constant, and the  $k^2$  in the denominator reflects the fact that we are considering a massless particle, as in QCD. Clearly the  $k^2 \rightarrow 0$  limit is relevant to the question whether the higher order graph can be neglected relative to the lower order one. Because one finds that for small  $k$  and large  $T$ ,  $\Pi_1(k)$  behaves as  $T^2$ , the ratio  $\Pi_2/\Pi_1$  is  $g^2 T^2/k^2$ . As a result when  $k$  is  $O(gT)$  or smaller the two-loop amplitude is not negligible compared to the one-loop amplitude. Thus graphs with “soft” external momenta [ $O(gT)$  or smaller] have to be included as insertions in higher order calculations.

A terminology has arisen: graphs with generic/soft external moment [ $O(gT)$  where  $g$  is small and  $T$  is large] and large internal momenta [the internal momenta are integration variables in an amplitude; when  $T$  is large they are  $O(T)$ , hence also large] are called “hard thermal loops.”[3, 4] Much study has been expended on them and finally a general picture has emerged. Before presenting general results, let us look at a specific example — a 2-point Green’s function.

It needs to be appreciated that in the imaginary-time formalism the correlation functions are unique and definite. But passage to real time, requires continuing from the integer-valued “energy” to a continuous variable, and this cannot be performed uniquely. This reflects the fact that in real time there exists a variety of correlation functions: time ordered products, retarded commutators, advanced commutators, *etc.* Essentially one is seeing the consequence of the fact that a Euclidean Laplacian possesses a unique inverse, whereas giving an inverse for the Minkowskian d’Alembertian requires specifying temporal boundary conditions, and a variety of answers can be gotten with a variety of boundary conditions.

Thus, when presenting results one needs to specify precisely what one is computing.

We shall consider a correlation function for two fermionic currents, in the 1-loop approximation.

$$\begin{aligned} \Pi^{\mu\nu}(x, y) &= -i \langle j^\mu(x) j^\nu(y) \rangle \\ &= \int \frac{d^4 k}{(2\pi)^4} e^{-ik(x-y)} \Pi^{\mu\nu}(k) \end{aligned}$$

The QCD result differs from the QED result by a group theoretical multiplicative factor, so we present high- $T$  results only for the latter, in real-time, and consider the time-ordained product  $\Pi_T^{\mu\nu}$  as well as the retarded commutator  $\Pi_R^{\mu\nu}$ .

$\Pi^{\mu\nu}$  possesses a real and an imaginary part. It is found that at large  $T$ , the real parts of  $\Pi_T^{\mu\nu}$  and  $\Pi_R^{\mu\nu}$  coincide.

$$-\text{Re}\Pi^{\mu\nu}(k) = \frac{T^2}{6}P_2^{\mu\nu} + \frac{T^2k^2}{|\mathbf{k}^2|} \left[ 1 + \frac{k^0}{2|\mathbf{k}|} \ln \left| \frac{k^0 - |\mathbf{k}|}{k^0 + |\mathbf{k}|} \right| \right] \left[ \frac{1}{3}P_1^{\mu\nu} + \frac{1}{2}P_2^{\mu\nu} \right]$$

where the projection operators are

$$\begin{aligned} P_1^{\mu\nu} &= g^{\mu\nu} - k^\mu k^\nu / k^2 \\ P_2^{\mu\nu} &= \begin{cases} 0 & \text{if } \mu \text{ or } \nu = 0 \\ \delta^{ij} - k^i k^j / |\mathbf{k}|^2 & \text{otherwise} \end{cases} \end{aligned}$$

For the imaginary part, which is present only for space-like arguments, different expressions are found.

$$\begin{aligned} -\text{Im}\Pi_R^{\mu\nu}(k) &\equiv -\pi\rho^{\mu\nu}(k) \\ &= \frac{\pi k^2}{|\mathbf{k}|^3} \frac{k^0}{2} T^2 \theta(-k^2) \left[ \frac{1}{3}P_1^{\mu\nu} + \frac{1}{2}P_2^{\mu\nu} \right] \\ -\text{Im}\Pi_T^{\mu\nu}(k) &= \frac{\pi k^2}{|\mathbf{k}|^3} \left[ \frac{k^0}{2} T^2 + T^3 \right] \theta(-k^2) \left[ \frac{1}{3}P_1^{\mu\nu} + \frac{1}{2}P_2^{\mu\nu} \right] \end{aligned}$$

A unified presentation of these formulas is achieved in a dispersive representation. For the retarded function this reads

$$\Pi_R^{\mu\nu}(k) = \Pi_{SUB}^{\mu\nu}(k) + \int dk'_0 \frac{\rho^{\mu\nu}(k'_0, \mathbf{k})}{k'_0 - k_0 - i\epsilon}$$

while the time-ordered expression is

$$\Pi_T^{\mu\nu}(k) = \Pi_{SUB}^{\mu\nu}(k) + \int dk'_0 \frac{\rho^{\mu\nu}(k'_0, \mathbf{k})}{k'_0 - k_0 - i\epsilon} + \frac{2\pi i}{e^{k_0/T} - 1} \rho^{\mu\nu}(k_0, \mathbf{k})$$

The dispersive expressions may also be used to give the imaginary-time formula.

$$\Pi_{\text{imaginary time}}^{\mu\nu}(k) = \Pi_{SUB}^{\mu\nu}(k) + \int dk'_0 \frac{\rho^{\mu\nu}(k'_0, \mathbf{k})}{k'_0 - 2\pi i n T}$$

In all the above formulas,  $\Pi_{SUB}^{\mu\nu}$  is a real subtraction term.

Note that a universal statement about high- $T$  behavior can be made only for the absorptive part  $\rho^{\mu\nu}$ : it is  $O(T^2)$ . This also characterizes  $\Pi_R^{\mu\nu}$ , but  $\Pi_T^{\mu\nu}$  possesses an additional  $O(T^3)$  imaginary part, which is seen to arise from the additional term in  $\Pi_T^{\mu\nu}$  involving the bosonic distribution function  $\frac{1}{e^{k_0/T} - 1}$ . Finally, the  $\Pi_{\text{imaginary time}}^{\mu\nu}$  amplitude has a temperature behavior determined by its external “energy”  $= 2\pi i n T$ . If this is replaced by a fixed  $k_0$  ( $T$ -independent) or if only the  $n = 0$  mode is considered, then one may assign an  $O(T^2)$  behavior to this quantity as well.

In conclusion, we assert that the 2-point correlation function behaves as  $O(T^2)$ , where it is understood that this statement is to be applied to the retarded amplitude, or to the imaginary time amplitude with its “energy” argument continued away from  $2\pi i n T$ . [5]

Similar analysis has been performed on the higher-point functions and this work has culminated with the discovery (Braaten, Pisarski, Frenkel, Taylor) [3, 4, 6] of a remarkable simplicity in their structure. To describe this simplicity, we do not discuss the individual  $n$ -point functions, but rather their sum multiplied by powers of the vector potential, *viz.* we consider the generating

functional for single-particle irreducible Green's functions with gauge field external lines in the hard thermal limit. (Effectively, we are dealing with continued imaginary-time amplitudes.) We call this quantity  $\Gamma_{\text{HTL}}(A)$  and it is computed in an  $SU(N)$  gauge theory containing  $N_F$  fermion species of the fundamental representation.  $\Gamma_{\text{HTL}}$  is found (i) to be proportional to  $(N + \frac{1}{2}N_F)$ , (ii) to behave as  $T^2$  at high temperature, and (iii) to be gauge invariant.

$$\begin{aligned}
& \text{A wavy line with a circle} + \text{A wavy line with a circle and one wavy line down} + \text{A wavy line with a circle and two wavy lines down} + \text{A wavy line with a circle and three wavy lines down} + \dots \\
& = (N + \frac{1}{2}N_F) \frac{g^2 T^2}{12\pi} \Gamma_{\text{HTL}}(A) \\
& \Gamma_{\text{HTL}}(U^{-1} A U + U^{-1} dU) = \Gamma_{\text{HTL}}(A)
\end{aligned}$$

(Henceforth  $g$ , the coupling constant, is scaled to unity.) A further kinematical simplification in  $\Gamma_{\text{HTL}}$  has also been established. To explain this we define two light-like four-vectors  $Q_{\pm}^{\mu}$  depending on a unit three-vector  $\hat{q}$ , pointing in an arbitrary direction.

$$Q_{\pm}^{\mu} = \frac{1}{\sqrt{2}}(1, \pm \hat{q})$$

$$\hat{q} \cdot \hat{q} = 1, \quad Q_{\pm}^{\mu} Q_{\pm\mu} = 0, \quad Q_{\pm}^{\mu} Q_{\mp\mu} = 1$$

Coordinates and potentials are projected onto  $Q_{\pm}^{\mu}$ .

$$x^{\pm} \equiv x_{\mu} Q_{\pm}^{\mu}, \quad \partial_{\pm} \equiv Q_{\pm}^{\mu} \frac{\partial}{\partial x^{\mu}}, \quad A_{\pm} \equiv A_{\mu} Q_{\pm}^{\mu}$$

The additional fact that is now known is that (iv) after separating an ultralocal contribution from  $\Gamma_{\text{HTL}}$ , the remainder may be written as an average over the angles of  $\hat{q}$  of a functional  $W$  that depends only on  $A_{+}$ ; also this functional is non-local only on the two-dimensional  $x^{\pm}$  plane, and is ultralocal in the remaining directions, perpendicular to the  $x^{\pm}$  plane. [“Ultralocal” means that any potentially non-local kernel  $k(x, y)$  is in fact a  $\delta$ -function of the difference  $k(x, y) \propto \delta(x - y)$ .]

$$\Gamma_{\text{HTL}}(A) = 2\pi \int d^4x A_0^a(x) A_0^a(x) + \int d\Omega_{\hat{q}} W(A_{+})$$

These results are established in perturbation theory, and a perturbative expansion of  $W(A_{+})$ , *i.e.* a power series in  $A_{+}$ , exhibits the above mentioned properties. A natural question is whether one can sum the series to obtain an expression for  $W(A_{+})$ .

Important progress on this problem was made when it was observed (Taylor, Wong)[6] that the gauge-invariance condition can be imposed infinitesimally, whereupon it leads to a functional differential equation for  $W(A_{+})$ , which is best presented as

$$\begin{aligned}
& \frac{\partial}{\partial x^{+}} \frac{\delta}{\delta A_{+}^a} \left[ W(A_{+}) + \frac{1}{2} \int d^4x A_{+}^b(x) A_{+}^b(x) \right] \\
& - \frac{\partial}{\partial x^{-}} [A_{+}^a] + f^{abc} A_{+}^b \frac{\delta}{\delta A_{+}^c} \left[ W(A_{+}) + \frac{1}{2} \int d^4x A_{+}^d(x) A_{+}^d(x) \right] = 0
\end{aligned}$$

In other words we seek a quantity, call it

$$S(A_+) \equiv W(A_+) + \frac{1}{2} \int d^4x A_+^a(x) A_+^a(x) ,$$

which is a functional on a two-dimensional manifold  $\{x^+, x^-\}$ , depends on a single functional variable  $A_+$ , and satisfies

$$\begin{aligned} \partial_1 \frac{\delta}{\delta A_1^a} S - \partial_2 A_1^a + f^{abc} A_1^b \frac{\delta}{\delta A_1^c} S &= 0 \\ \text{"1"} \equiv x^+ , \quad \text{"2"} \equiv -x^- , \quad A_1^a \equiv A_+^a \end{aligned}$$

Another suggestive version of the above is gotten by defining  $A_2^a \equiv \frac{\delta S}{\delta A_1^a}$ .

$$\partial_1 A_2^a - \partial_2 A_1^a + f^{abc} A_1^b A_2^c = 0$$

To solve the functional equation and produce an expression for  $W(A_+)$ , we now turn to a completely different corner of physics, and that is Chern-Simons theory at zero temperature.

The Chern-Simons term is a peculiar gauge theoretic topological structure that can be constructed in odd dimensions, and here we consider it in 3-dimensional space-time.

$$I_{\text{CS}} \propto \int d^3x \epsilon^{\alpha\beta\gamma} \text{Tr} (\partial_\alpha A_\beta A_\gamma + \frac{2}{3} A_\alpha A_\beta A_\gamma)$$

This object was introduced into physics over a decade ago, and since that time it has been put to various physical and mathematical uses. Indeed one of our originally stated motivations for studying the Chern-Simons term was its possible relevance to high-temperature gauge theory.[7] Here following Efraty and Nair,[8] we shall employ the Chern-Simons term for a determination of the hard thermal loop generating functional,  $\Gamma_{\text{HTL}}$ .

Since it is the space-time integral of a density,  $I_{\text{CS}}$  may be viewed as the action for a quantum field theory in (2+1)-dimensional space-time, and the corresponding Lagrangian would then be given by a two-dimensional, spatial integral of a Lagrange density.

$$\begin{aligned} I_{\text{CS}} &\propto \int dt L_{\text{CS}} \\ L_{\text{CS}} &\propto \int d^2x \left( A_2^a \dot{A}_1^a + A_0^a F_{12}^a \right) \end{aligned}$$

I have separated the temporal index (0) from the two spatial ones (1,2) and have indicated time differentiation by an over dot.  $F_{12}^a$  is the non-Abelian field strength, defined on a two-dimensional plane.

$$F_{12}^a = \partial_1 A_2^a - \partial_2 A_1^a + f^{abc} A_1^b A_2^c$$

Examining the Lagrangian, we see that it has the form

$$L \sim p\dot{q} - \lambda H(p, q)$$

where  $A_2^a$  plays the role of  $p$ ,  $A_1^a$  that of  $q$ ,  $F_{12}^a$  is like a Hamiltonian and  $A_0^a$  acts like the Lagrange multiplier  $\lambda$ , which forces the Hamiltonian to vanish; here  $A_0^a$  enforces the vanishing of  $F_{12}^a$ .

$$F_{12}^a = 0$$

The analogy instructs us how the Chern-Simons theory should be quantized.

We postulate equal-time commutation relations, like those between  $p$  and  $q$ .

$$\left[ A_1^a(\mathbf{r}), A_2^b(\mathbf{r}') \right] = i \delta^{ab} \delta(\mathbf{r} - \mathbf{r}')$$

In order to satisfy the condition enforced by the Lagrange multiplier, we demand that  $F_{12}^a$ , operating on “allowed” states, annihilate them.

$$F_{12}^a | \rangle = 0$$

This equation can be explicitly presented in a Schrödinger-like representation for the Chern-Simons quantum field theory, where the state is a functional of  $A_1^a$ . The action of the operators  $A_1^a$  and  $A_2^a$  is by multiplication and functional differentiation, respectively.

$$\begin{aligned} | \rangle &\sim \Psi(A_1^a) \\ A_1^a | \rangle &\sim A_1^a \Psi(A_1^a) \\ A_2^a | \rangle &\sim \frac{1}{i} \frac{\delta}{\delta A_1^a} \Psi(A_1^a) \end{aligned}$$

This, of course, is just the field theoretic analog of the quantum mechanical situation where states are functions of  $q$ , the  $q$  operator acts by multiplication, and the  $p$  operator by differentiation. In the Schrödinger representation, the condition that states be annihilated by  $F_{12}^a$

$$\left( \partial_1 A_2^a - \partial_2 A_1^a + f_{abc} A_1^b A_2^c \right) | \rangle = 0$$

leads to a functional differential equation.

$$\left( \partial_1 \frac{1}{i} \frac{\delta}{\delta A_1^a} - \partial_2 A_1^a + f_{abc} A_1^b \frac{1}{i} \frac{\delta}{\delta A_1^c} \right) \Psi(A_1^a) = 0$$

If we define  $S$  by  $\Psi = e^{iS}$  we get equivalently

$$\partial_1 \frac{\delta}{\delta A_1^a} S - \partial_2 A_1^a + f_{abc} A_1^b \frac{\delta}{\delta A_1^c} S = 0$$

This equation comprises the entire content of Chern-Simons quantum field theory.  $S$  is the Chern-Simons eikonal, which gives the exact wave functional owing to the simple dynamics of the theory. Also the above eikonal equation is recognized to be precisely the equation for the hard thermal loop generating functional, given above.

Let me elaborate on the connection with eikonal-WKB ideas. Let us recall that in particle quantum mechanics, when the wave function  $\psi(q)$  is written in eikonal form

$$\psi(q) = e^{iS(q)}$$

then the WKB approximation to  $S(q)$  is given by the integral of the canonical 1-form  $p dq$

$$S(q) = \int^q p(q') dq'$$

where  $p(q)$ , the momentum, is taken to be function of the coordinate  $q$ , by virtue of satisfying the equation of motion.

$$\begin{aligned} \frac{p^2(q)}{2} + V(q) &= E \\ p(q) &= \sqrt{2E - 2V(q)} \end{aligned}$$

Analogously, in the present field theory application, the eikonal  $S(A_1)$  may be written as

$$S(A_1) = \int^{A_1} A_2^a(A_1') \mathcal{D}A_1'^a$$

where  $A_2^a(A_1)$  is functional of  $A_1$  determined by the equation of motion

$$\partial_1 A_2^a - \partial_2 A_1^a + f^{abc} A_1^b A_2^c = 0$$

Since, by construction  $\frac{\delta S}{\delta A_1^a} = A_2^a$ , it is clear that as a consequence  $S$  satisfies the required equation. However, we reiterate that here there is no WKB approximation: everything is exact.

The gained advantage for thermal physics is that “acceptable” Chern-Simons states, *i.e.* solutions to the above functional equations, were constructed long ago,[9] and one can now take over those results to the hard thermal loop problem. One knows from the Chern-Simons work that  $\Psi$  and  $S$  are given by a 2-dimensional fermionic determinant, *i.e.* by the Polyakov-Wiegman expression. While these are not described by very explicit formulas, many properties are understood, and the hope is that one can use these properties to obtain further information about high-temperature QCD processes.

For example one can give a very explicit series expansion for  $\Gamma_{\text{HTL}}$  in terms of powers of  $A$

$$\Gamma_{\text{HTL}} = \frac{1}{2!} \int \Gamma_{\text{HTL}}^{(2)} AA + \frac{1}{3!} \int \Gamma_{\text{HTL}}^{(3)} AAA + \dots$$

where the non-local kernels  $\Gamma_{\text{HTL}}^{(i)}$  are known explicitly. This power series may be used to systematize the resummation procedure for perturbative theory. Here is what one does: perturbation theory for Green’s functions may be organized with the help of a functional integral, where the integrand contains (among other factors)  $e^{iI_{\text{QCD}}(A)}$  where  $I_{\text{QCD}}$  is the QCD action. We now rewrite that as

$$e^{i\left\{I_{\text{QCD}}(A) + \frac{m^2}{4\pi}\Gamma_{\text{HTL}}(A) - \frac{m^2}{4\pi}\Gamma_{\text{HTL}}(A)\right\}}$$

where  $m = T\sqrt{\frac{N+N_F/2}{3}}$ . Obviously nothing has changed, because we have merely added and subtracted the hard-thermal-loop generating functional. Next we introduce a loop counting parameter  $l$ : in an  $l$ -expansion, different powers of  $l$  correspond to different numbers of loops, but at the end  $l$  is set to unity. The resummed action is then taken to be

$$e^{iI_{\text{resummed}}} = e^{i\left\{\frac{1}{i^2}\left[I_{\text{QCD}}(lA) + \frac{m^2}{4\pi}\Gamma_{\text{HTL}}(lA)\right] - \frac{m^2}{4\pi}\Gamma_{\text{HTL}}(lA)\right\}}$$

One readily verifies that an expansion in powers of  $l$  describes the resummed perturbation theory, free of infrared divergences.

Even though the closed form for  $\Gamma_{\text{HTL}}$  is not very explicit, a much more explicit formula can be gotten for its functional derivative  $\frac{\delta\Gamma_{\text{HTL}}}{\delta A_\mu^a}$ . This may be identified with an induced current, which is then used as a source in the Yang-Mills equation. Thereby one obtains a non-Abelian generalization of the Kubo equation, which governs the response of the hot quark gluon plasma to external disturbances.[5]

$$D_\mu F^{\mu\nu} = \frac{m^2}{2} j_{\text{induced}}^\nu$$

From the known properties of the fermionic determinant — hard thermal loop generating functional — one can show that  $j_{\text{induced}}^\mu$  is given by

$$j_{\text{induced}}^\mu = \int \frac{d\Omega_{\hat{q}}}{4\pi} \left\{ Q_+^\mu \left( a_-(x) - A_-(x) \right) + Q_-^\mu \left( a_+(x) - A_+(x) \right) \right\}$$



where  $a_{\pm}$  are solutions to the equations

$$\begin{aligned}\partial_+ a_- - \partial_- A_+ + [A_+, a_-] &= 0 \\ \partial_+ A_- - \partial_- a_+ + [a_+, A_-] &= 0\end{aligned}$$

Evidently  $j_{\text{induced}}^{\mu}$ , as determined by the above equations, is a non-local and non-linear functional of the vector potential  $A_{\mu}$ .

There now have appeared several alternative derivations of the Kubo equation. Blaizot and Iancu[10] have analyzed the Schwinger-Dyson equations in the hard thermal regime; they truncated them at the 1-loop level, made further kinematical approximations that are justified in the hard thermal limit, and they too arrived at the Kubo equation. Equivalently the argument may be presented succinctly in the language of the composite effective action,[11] which is truncated at the 1-loop (semi-classical) level — two-particle irreducible graphs are omitted. The stationarity condition on the 1-loop action is the gauge invariance constraint on  $\Gamma_{\text{HTL}}$ . Finally, there is one more, entirely different derivation — which perhaps is the most interesting because it relies on classical physics.[12] We shall give the argument presently, but first we discuss solutions for the Kubo equation.

To solve the Kubo equation, one must determine  $a_{\pm}$  for arbitrary  $A_{\pm}$ , thereby obtaining an expression for the induced current, as a functional of  $A_{\pm}$ . Since the functional is non-local and non-linear, it does not appear possible to construct it explicitly in all generality. However, special cases can be readily handled.

In the Abelian case, everything commutes and linearizes. One can determine  $a_{\pm}$  in terms of  $A_{\pm}$ .

$$a_{\pm} = \frac{\partial_{\pm}}{\partial_{\mp}} A_{\mp}$$

Incidentally, this formula exemplifies the kinematical simplicity, mentioned above, of hard thermal loops: the nonlocality of  $1/\partial_{\pm}$  lies entirely in the  $\{x^+, x^-\}$  plane. With the above form for  $a_{\pm}$  inserted into the Kubo equation, the solution can be constructed explicitly. It coincides with the results obtained by Silin long ago, on the basis of the Boltzmann-Vlasov equation.[13] One sees that the present theory is the non-Abelian generalization of that physics; in particular  $m$ , given above, is recognized as the Debye screening length, which remains gauge invariant in the non-Abelian context.

It is especially interesting to emphasize that Silin did not use quantum field theory in his derivation; rather he employed classical transport theory. Nevertheless, his final result coincides with what here has been developed from a quantal framework. This raises the possibility that the non-Abelian Kubo equation can also be derived classically, and indeed such a derivation has been given, as mentioned above.

We now pause in our discussion of solutions to the non-Abelian Kubo equation in order to describe its classical derivation.

Transport theory is formulated in terms of a single-particle distribution function  $f$  on phase space. In the Abelian case,  $f$  depends on position  $\{x^{\mu}\}$  and momentum  $\{p^{\mu}\}$  of the particle. For the non-Abelian theory it is necessary to take into account the fact that the particle's non-Abelian charge  $\{Q^a\}$  also is a dynamical variable:  $Q^a$  satisfies an evolution equation (see below) and is an element of phase space. Therefore, the non-Abelian distribution function depends on  $\{x^{\mu}\}$ ,  $\{p^{\mu}\}$  and  $\{Q^a\}$ , and in the collisionless approximation obeys the transport equation  $\frac{d}{d\tau}f = 0$ , *i.e.*

$$\frac{\partial f}{\partial x^{\mu}} \frac{dx^{\mu}}{d\tau} + \frac{\partial f}{\partial p^{\mu}} \frac{dp^{\mu}}{d\tau} + \frac{\partial f}{\partial Q^a} \frac{dQ^a}{d\tau} = 0$$

The derivatives of the phase-space variables are given by the Wong equations, for a particle with mass  $\mu$ .

$$\begin{aligned}\frac{dx^\mu}{d\tau} &= \frac{p^\mu}{\mu} \\ \frac{dp^\mu}{d\tau} &= F_a^{\mu\nu} \frac{dx_\nu}{d\tau} Q^a \\ \frac{dQ^a}{d\tau} &= -f^{abc} \frac{dx^\mu}{d\tau} A_\mu^b Q^c\end{aligned}$$

In order to close the system we need an equation for  $F^{\mu\nu}$ . In a microscopic description (with a single particle) one would have  $(D_\mu F^{\mu\nu})^a = \int d\tau Q^a(\tau) \frac{p^\nu(\tau)}{\mu} \delta^4(x - x(\tau))$  and consistency would require covariant conservation of the current; this is ensured provided  $Q^a$  satisfies the equation given above. In our macroscopic, statistical derivation, the current is given in terms of the distribution function, so the system of equations closes with

$$(D_\mu F^{\mu\nu})^a = \int dp dQ Q^a p^\nu f(x, p, Q)$$

The collisionless transport equation, with the equations of motion inserted, is called the Boltzmann equation. The closed system formed by the latter supplemented with the Yang-Mills equation is known as the non-Abelian Vlasov equations. To make progress, this highly non-linear set of equations is approximated by expanding around the equilibrium form for  $f$ ,

$$f_{\text{boson/fermion}}^{\text{free}} \propto \left( e^{\frac{1}{T} \sqrt{\mathbf{p}^2 + \mu^2}} \mp 1 \right)^{-1}$$

This comprises the Vlasov approximation, and readily leads to the non-Abelian Kubo equation.[12]

One may say that the non-Abelian theory is the minimal elaboration of the Abelian case needed to preserve non-Abelian gauge invariance. The fact that classical reasoning can reproduce quantal results is presumably related to the fact that the quantum theory makes use of the (resummed) 1-loop approximation, which is frequently reorganized as an essentially classical effect. Evidently, the quantum fluctuations included in the hard thermal loops coincide with thermal fluctuations.

Returning now to our summary of the solutions to the non-Abelian Kubo equation that have been obtained thus far, we mention first that the static problem may be solved completely.[11] When the *Ansatz* is made that the vector potential is time independent,  $A_\pm = A_\pm(\mathbf{r})$ , one may solve for  $a_\pm$  to find  $a_\pm = -A_\pm$  and the induced current is explicitly computed as

$$\frac{m^2}{2} j_{\text{induced}}^\mu = \begin{pmatrix} -m^2 A^0 \\ \mathbf{0} \end{pmatrix}$$

This exhibits gauge-invariant electric screening with Debye mass  $m$ . One may also search for localized static solutions to the Kubo equation, but one finds only infinite energy solutions, carrying a point-magnetic monopole singularity at the origin. Thus there are no plasma solitons in high-T QCD.[11] Specifically, one finds that upon selecting the radially symmetric solution, which decreases at large distances, there arises a magnetic monopole-like singularity at the origin.

Much less is known concerning time-dependent solutions. Blaizot and Iancu[14] have made the *Ansatz* that the vector potentials depend only on the combination  $x \cdot k$ , where  $k$  is an arbitrary 4-vector:  $A_\pm = A_\pm(x \cdot k)$ . Once again  $a_\pm$  can be determined; one finds  $a_\pm = \frac{Q_\pm \cdot k}{Q_\mp \cdot k} A_\mp$ , and the induced

current is computable. For  $k = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ , where there is no space dependence (only a dependence on time is present) one finds

$$\frac{m^2}{2} j_{\text{induced}}^\mu = \begin{pmatrix} 0 \\ -\frac{1}{3} m^2 \mathbf{A} \end{pmatrix}$$

More complicated expressions hold with general  $k$ . The Kubo equation can be solved numerically; the resulting profile is a non-Abelian generalization of a plasma plane wave.

The physics of all these solutions, as well as of other, still undiscovered ones, remains to be elucidated, and I invite any of you to join in this interesting task.

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